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GROUP RINGS OF HYPERABELIAN GROUPS

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ABSTRACT

If k is a field of characteristic $p > 0$ then we find all hyper-(Abelian or locally finite-p') groups G such that the augmentation ideal of the group algebra kG has the AR property.

Let $\mathscr{X}$ be a class of groups. A group G is a hyper-$\mathscr{X}$ group if every non-trivial homomorphic image of G contains a non-trivial normal subgroup which is an $\mathscr{X}$-group. If $\mathscr{Y}$ is also a class of groups then by an $\mathscr{X}\mathscr{Y}$-group we shall mean a group G which contains a normal subgroup H such that H is an $\mathscr{X}$-group and G/H a $\mathscr{Y}$-group. If $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_n$ is a finite collection of group classes then by an $\mathscr{X}_1\mathscr{X}_2\cdots\mathscr{X}_n$-group is meant an $\mathscr{X}_1(\mathscr{X}_2(\mathscr{X}_3\cdots(\mathscr{X}_{n-1}\mathscr{X}_n)))$-group. We shall denote the classes of Abelian, finitely generated and finite groups by $\mathscr{A}$, $\mathscr{G}$, $\mathscr{F}$, respectively. Let p be a prime. Then $\mathscr{N}_p$ will denote the class of nilpotent groups such that each upper central factor is an extension of a finitely generated group by a p'-torsion group. (A p'-torsion group is a torsion group such that every non-trivial element has order coprime to p.) A group G is locally finite-p' if every finite subset generates a finite subgroup of order coprime to p. The class of locally finite-p' groups will be denoted by $\mathscr{L}_p$.

If G is a group then a p-chief factor of G is a group H/K such that $K \subseteq H$, K and H are normal subgroups of G and H/K is a minimal normal p-subgroup of the group G/K. If k is a field then $\mathfrak{g}_k$, or $\mathfrak{g}$ if there is no ambiguity about k, will denote the augmentation ideal of the group algebra kG. The purpose of this note is to prove the following theorem which generalizes [4, theorem E].

THEOREM A. *Let k be a field of characteristic $p > 0$ and G a hyper-$(\mathscr{A} \cup \mathscr{L}_p)$ group. Then the augmentation ideal of the group algebra kG has the* AR *property if and only if G is an $\mathscr{L}_p\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group and G centralizes all p-chief factors.*

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The ideal $\mathfrak{g}$ has the AR property provided for every right ideal E and left ideal L of $R = kG$ there exists a positive integer n such that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g}$ and $L \cap \mathfrak{g}^n \leqq \mathfrak{g}L$. Let $\varphi: R \to R$ be the antiautomorphism induced by the anti-automorphism $x \to x^{-1}$ $(x \in G)$ of G. Since $\varphi(\mathfrak{g}) = \mathfrak{g}$ it follows that $\mathfrak{g}$ has the AR property if and only if for every right ideal E there exists $n \geqq 1$ such that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g}$.

Theorem A is a consequence of the following result.

THEOREM B. *Let G be a group which contains a finite chain $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ of normal subgroups G_i such that G_i/G_{i-1} is an $\mathscr{L}_p$-group or an $(\mathscr{A} \cap \mathscr{N}_p)$-group for each $1 \leqq i \leqq n$. Then G is an $\mathscr{L}_p\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group.*

1. Proof of Theorem A

In this section we show how to derive Theorem A from Theorem B. Let p be a prime. Following [4] we say that a group G satisfies Max-sn$_p^*$ if for any ascending chain of subgroups $H_1 \leqq H_2 \leqq H_3 \leqq \cdots$ such that H_i is normal in H_{i+1} for each $i \geqq 1$, there exists a positive integer n_0 such that H_{n+1}/H_n is an $\mathscr{L}_p$-group for all $n \geqq n_0$.

PROOF OF THEOREM A. Let k be a field of characteristic $p > 0$ and G a hyper-$(\mathscr{A} \cup \mathscr{L}_p)$ group. Suppose first that the augmentation ideal $\mathfrak{g}$ of kG has the AR property. By [4, lemma A] G satisfies Max-sn$_p^*$ and by [4, lemma B] there exists a finite chain

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of normal subgroups G_i such that G_i/G_{i-1} is an $\mathscr{A}$-group or an $\mathscr{L}_p$-group for each $1 \leqq i \leqq n$. Since G satisfies Max-sn$_p^*$ it follows that each Abelian factor G_i/G_{i-1} is an extension of a finitely generated group by a p'-torsion group, i.e. G_i/G_{i-1} is an $(\mathscr{A} \cap \mathscr{N}_p)$-group. Also, by [4, lemma 3.3] G centralizes all p-chief factors. Hence by Theorem B, G is an $\mathscr{L}_p\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group.

Conversely, suppose that G is an $\mathscr{L}_p\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group. Let L be a normal $\mathscr{L}_p$-subgroup of G such that $H = G/L$ is an $\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group. Suppose that $\mathfrak{h}$ has the AR property. If E is a right ideal of kG then there exists $n \geqq 1$ such that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g} + \bar{\mathfrak{l}}$ where $\bar{\mathfrak{l}} = \mathfrak{l}(kG)$ and $\mathfrak{l}$ is the augmentation ideal of kL. Since L is an $\mathscr{L}_p$-group it follows that kL is Von Neumann regular and hence for each element r in $\bar{\mathfrak{l}}$ there exists a in $\mathfrak{g}$ such that $r(1-a) = 0$. It follows that $E \cap \mathfrak{g}^n \leqq E\mathfrak{g}$. Thus, without loss of generality $L = 1$.

Let N be a normal $\mathscr{N}_p$-subgroup of G such that G/N is a $(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group. By [3, theorem C′] the augmentation ideal $\mathfrak{n}$ of kN has the AR property. Let

$S = kN$. Then S satisfies the right and left Ore condition with respect to $T = \{1 - a\colon a \in \mathfrak{n}\}$. Let $R = kG$. Because T is G-invariant, R satisfies the right and left Ore conditions with respect to T (see the proof of [1, lemma 13.3.5 (ii)]). If

$$I = \{s \in S\colon st = 0 \quad \text{for some} \quad t \text{ in } T\}$$

and

$$J = \{r \in R\colon rt = 0 \quad \text{for some} \quad t \text{ in } T\}$$

then I is a G-invariant ideal of S and $J = IR = RI$. The elements $t + I(t \in T)$ of S/I are all regular and hence the elements $t + J$ $(t \in T)$ of R/J are regular. We can form the partial quotient rings $S_{\mathfrak{n}}$ and $R_{\mathfrak{n}}$ where $R_{\mathfrak{n}}$ is the partial quotient ring of R with respect to T. Note that $S_{\mathfrak{n}}$ is a subring of $R_{\mathfrak{n}}$. By [3, corollary C1] or [5, theorem B] $S_{\mathfrak{n}}$ is a Noetherian ring and clearly $\mathfrak{n}\, S_{\mathfrak{n}}$ has a centralizing set of generators. By a result of Roseblade, $\mathfrak{n}R_{\mathfrak{n}}$ has the AR property and hence $\bar{\mathfrak{n}} = \mathfrak{n}R$ has the AR property in the ring R [1, theorem 11.2.9].

Let F be a right ideal of R. There exists a positive integer m such that $F \cap \bar{\mathfrak{n}}^m \subseteq F\bar{\mathfrak{n}}$. Let $X = N^{p^m}$. Then N/X is an $\mathcal{N}_p$-group and also a p-group. It follows that N/X is finite. Let $Q = G/X$. Then Q is polycyclic-by-finite and Q centralizes p-chief factors so that by another result of Roseblade $\mathfrak{q}$ has the AR property (see [1, theorem 11.2.15]). There exists a positive integer d such that $F \cap \mathfrak{g}^d \subseteq F\mathfrak{g} + \bar{\mathfrak{x}}$. But $\bar{\mathfrak{x}} \subseteq \bar{\mathfrak{n}}^m$ and it follows that $F \cap \mathfrak{g}^d \subseteq F\mathfrak{g}$. Thus $\mathfrak{g}$ has the AR property and this completes the proof of Theorem A.

2. Proof of Theorem B

Let $\mathscr{X}$ denote the class of groups G such that there exists a finite chain

$$(*) \qquad 1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of normal subgroups G_i such that G_i/G_{i-1} is an $\mathscr{L}_p$-group or an $(\mathscr{A} \cap \mathcal{N}_p)$ group for each $1 \leqq i \leqq n$. Let $\mathscr{X}_p$ denote the class of $\mathscr{X}$-groups G which centralize all p-chief factors.

Lemma 1 (See [4, lemma F]). $\mathscr{X} \subseteq \mathscr{X}_p F$.

Lemma 2. *If G is a torsion $\mathscr{X}_p$-group then G is an extension of an $\mathscr{L}_p$-group by a finite p-group.*

Proof. Let L be the maximal normal $\mathscr{L}_p$-subgroup of G (see [2, p. 35 theorem 1.45]). Suppose $L \neq G$ and G/L is not a finite p-group. Then there exist normal subgroups H and K of G such that $L \subseteq K \subseteq H$, K/L is a non-trivial finite p-group and H/K a non-trivial $\mathscr{L}_p$-group. Without loss of generality we can suppose that $G = H$. Choose $T \lhd G$ such that $L \subseteq T \subseteq K$ and K/T is a p-chief factor of G. Then K/T is central in G/T and by a theorem of Schur (see [2, p. 102 theorem 4.12]) $(G/T)'$, the derived subgroup of G/T, is an $\mathscr{L}_p$-group. Thus there exists $X \lhd G$ such that $T \subseteq X$, X/T is an $\mathscr{L}_p$-group and G/X a finite p-group. Now consider X/L. By induction on the order of K/L it follows that X/L is an extension of an $\mathscr{L}_p$-group by a finite p-group. Thus G/L is a finite p-group by [2, p. 35 theorem 1.45], a contradiction. The result follows.

Let $\mathscr{F}_p$ denote the class of finite p-groups. Then Lemma 2 states that torsion $\mathscr{X}_p$-groups are $\mathscr{L}_p\mathscr{F}_p$-groups.

Lemma 3. *If an $\mathscr{X}_p$-group G is an extension of a torsion-free Abelian group by an $\mathscr{L}_p$-group then G is an $\mathscr{L}_p\mathscr{A}$-group.*

Proof. Let A be a torsion-free Abelian normal subgroup of G such that G/A is an $\mathscr{L}_p$-group. Let n be a positive integer. Then G/A^{p^n} is a torsion $\mathscr{X}_p$-group and hence there exists $H \lhd G$ such that $A^{p^n} \leqq H$, H/A^{p^n} is an $\mathscr{L}_p$-group and G/H an $\mathscr{F}_p$-group (Lemma 2). Then $G = HA$ and $H \cap A = A^{p^n}$. Since $G/H \cong A/A^{p^n}$ it follows that G/H is Abelian and hence $G' \subseteq H$. Thus $A \cap G' \subseteq A^{p^n}$ for all positive integers n. There exists a finitely generated subgroup B of A such that A/B is a p'-group. Then $A^{p^n} \cap B = B^{p^n}$ and hence $(\bigcap_{n=1}^{\infty} A^{p^n}) \cap B = 1$. Since A is torsion-free it follows that $\bigcap_{n=1}^{\infty} A^{p^n} = 1$ and hence $A \cap G' = 1$. Thus G' is an $\mathscr{L}_p$-group and it follows that G is an $\mathscr{L}_p\mathscr{A}$-group.

The next lemma is clear.

Lemma 4. $(\mathscr{F}_p\mathscr{N}_p) \cap \mathscr{X}_p \leqq \mathscr{N}_p$.

A group G has finite Prufer rank r, where r is a positive integer, if every finitely generated subgroup can be generated by r elements, and r is the least positive integer with this property.

Lemma 5. $\mathscr{X}_p \leqq \mathscr{L}_p\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$.

Proof. Let G be an $\mathscr{X}_p$-group with series $(*)$. We prove the result by induction on n, the case $n = 1$ being clear. Suppose first that G does not contain any non-trivial normal $\mathscr{L}_p$-subgroup. Let $A = G_1$. Then A is an Abelian subgroup of G. Let $L \lhd G$ with $A \subseteq L$ such that L/A is the maximal normal $\mathscr{L}_p$-subgroup of G/A. By induction on n the group G/L is an $\mathscr{N}_p(\mathscr{G} \cap \mathscr{A})\mathscr{F}$-group.

Suppose first that A is a torsion group. Then A is a finite p-group. By Lemma 2, $L = A$ and by Lemma 4, G is an $\mathcal{N}_p(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group. So suppose that A is not a torsion group. Let T be the torsion subgroup of A. Then T is a finite p-group and so there exists a positive integer n such that A^{p^n} is torsion-free. Moreover, A/A^{p^n} is a finite p-group since G is an $\mathcal{X}_p$-group. Thus G/A^{p^n} is an $\mathcal{L}_p\mathcal{N}_p(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group by Lemmas 2 and 4. So without loss of generality we can suppose that A is torsion-free.

Let C be the centralizer in L of A. Then L/C is Q-linear and so finite (see [6, theorem 9.33]). Moreover, $A \subseteq C$ and C/A is an $\mathcal{L}_p$-group. By Lemma 3, C is Abelian. The torsion subgroup of C is a finite p-group and hence there exists a positive integer m such that $H = C^{p^m}$ is torsion-free and C/H is a finite p-group. Thus L/H is a finite group and by Lemma 2, L/H is an $\mathcal{L}_p\mathcal{F}_p$-group. Applying Lemma 3, we see that L is an $\mathcal{A}\mathcal{F}_p$-group. As before, L contains a normal subgroup K of G such that K is torsion-free Abelian and L/K is an $\mathcal{F}_p$-group.

Let $N \lhd G$ such that $L \subseteq N$ and N/L is the Fitting subgroup of G/L. Then the torsion subgroup of N/L is a finite p-group. Since G is an $\mathcal{X}_p$-group it follows that N/K is a nilpotent group and the torsion subgroup of N/K is a finite p-group. There exists a positive integer s such that if $U = N^{p^s}K$ then U/K is torsion-free and N/U is a p-group. Since G is an $\mathcal{X}_p$-group it follows that N/U is a finite group. Since G/U is a $(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group we know that G/U is a $(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group. In fact there exists a normal subgroup V of G such that $U \subseteq V$, V/U is finitely generated torsion-free Abelian and G/V is finite. Thus V is torsion-free solvable and G/V finite. Moreover, G has finite Prufer rank by [2, p. 34 lemma 1.44]. By [7, theorem 1.1] G is Q-linear. By Mal'cev's Theorem (see [6, theorem 3.6]) G is an $\mathcal{N}_p(\mathcal{A} \cap \mathcal{N}_p)\mathcal{F}$-group. Moreover, there exists a finite algebraic extension K of Q and normal subgroups $X \subseteq Y$ of G such that X is an $\mathcal{N}_p$-group, G/Y a finite group and Y/X an $(\mathcal{A} \cap \mathcal{N}_p)$-group such that Y/X can be embedded in the direct product of a finite number of copies of K^*, the multiplicative group of non-zero elements of K. It follows that Y/X is finitely generated and hence G is an $\mathcal{N}_p(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group.

Finally, let G be an $\mathcal{X}_p$-group. Let M be the maximal normal $\mathcal{L}_p$-subgroup of G. Then G/M does not contain any non-trivial normal $\mathcal{L}_p$-subgroups by [2, p. 35 theorem 1.45] and by the above argument G/M is an $\mathcal{N}_p(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group. Thus G is an $\mathcal{L}_p\mathcal{N}_p(\mathcal{G} \cap \mathcal{A})\mathcal{F}$-group.

Combining Lemmas 1 and 5 completes the proof of Theorem B.

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